2025 SPHERIC World Conference

An enhanced SPH-based hydroelastic FSI solver with structural dynamic hourglass control

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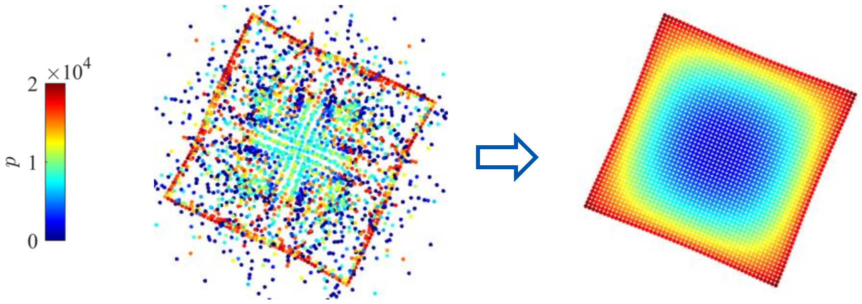
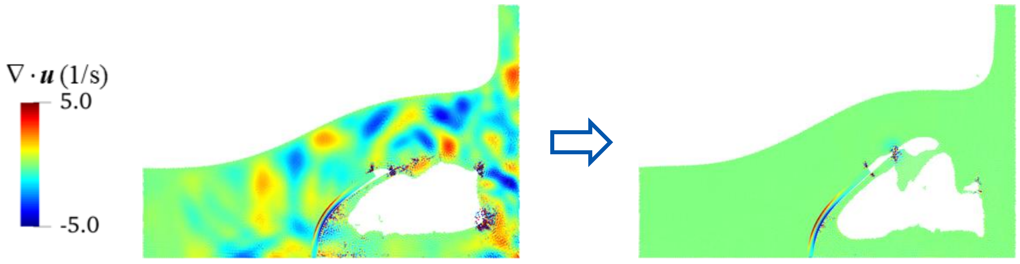
17th June 2025





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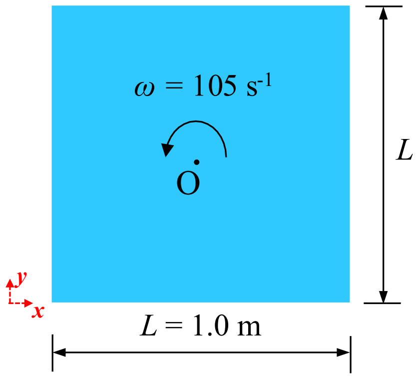
1 Motivation

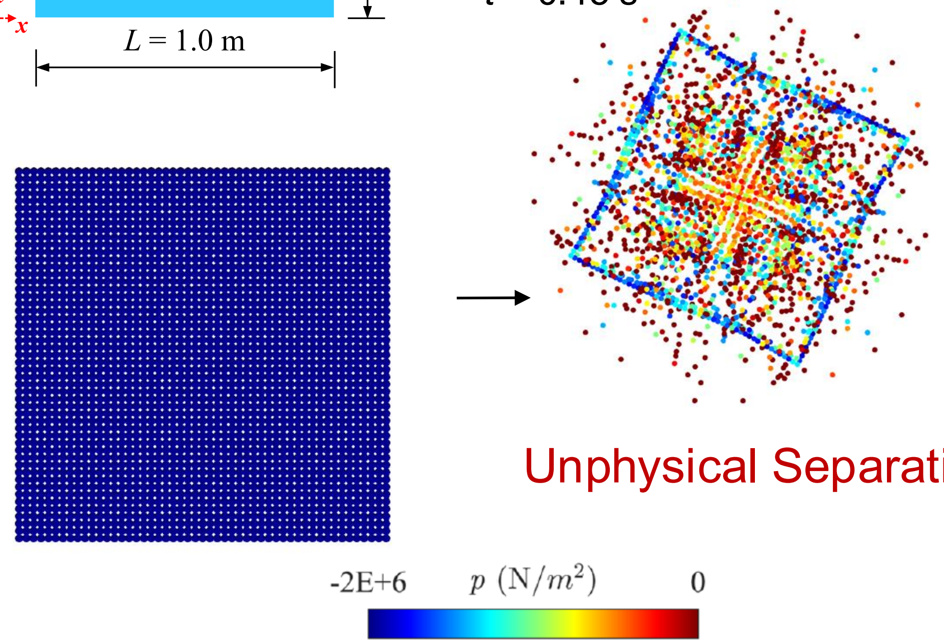
➢ Low solution accuracy



High-speed rotation of elastic plate

$$\frac{D\boldsymbol{u}_i}{Dt} = \sum_j (\boldsymbol{P}_i L_i + \boldsymbol{P}_j L_j) \nabla_i W_{ij} V_i^0 V_j^0$$

t = 0.15 s



Unphysical Separation

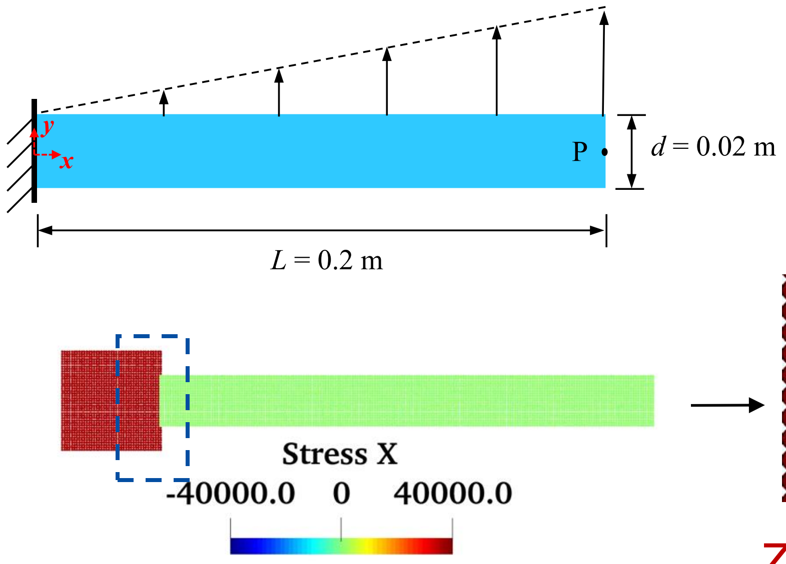
Simulated by Total Lagrangian SPH

- Low solution accuracy leads to non-physical results
- Rank deficiency / Hourglass mode

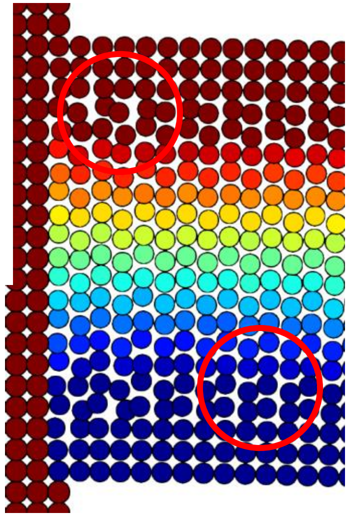
1 Motivation

➢ Serious hourglass modes

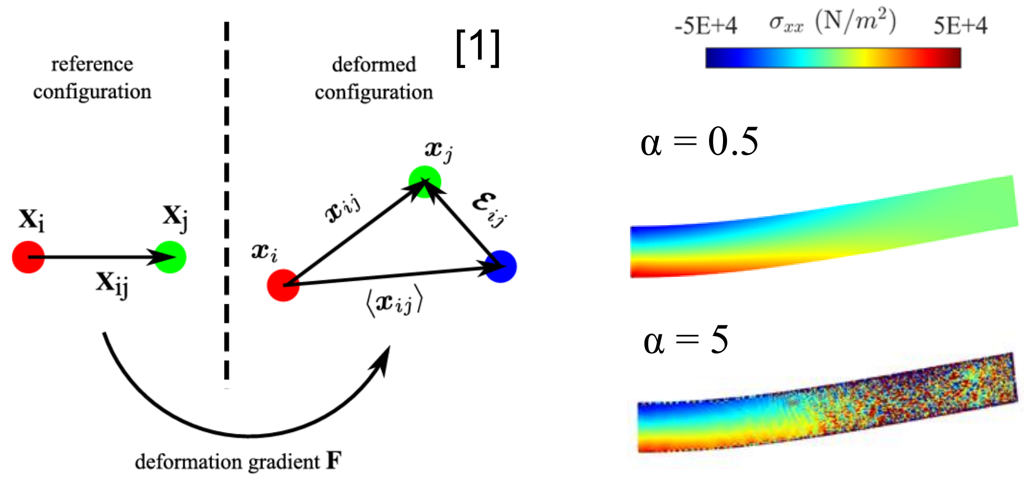
Free oscillation of a cantilever plate



Unphysical clustering



Zigzag distribution



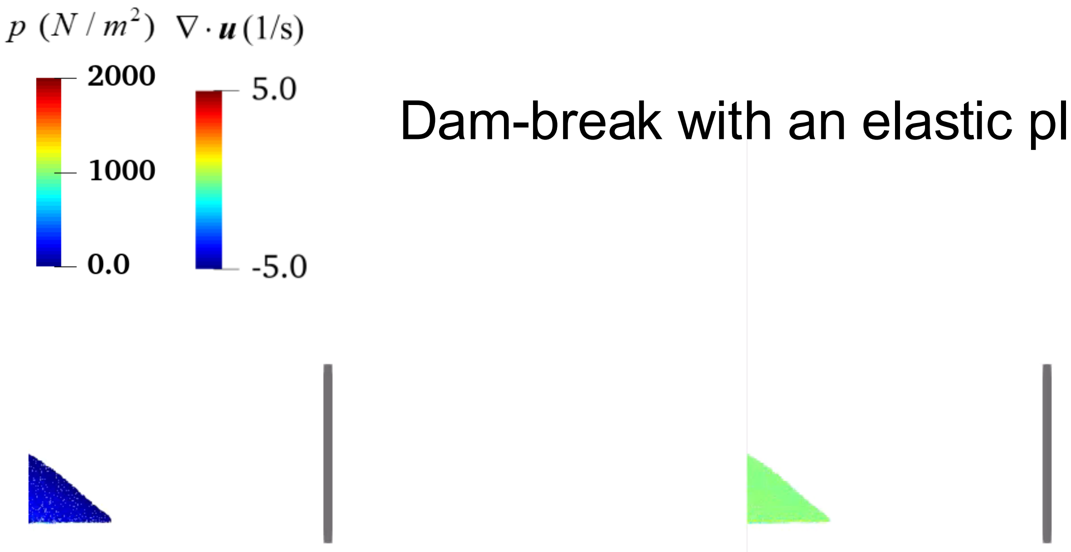
$$f_i^{\mathrm{HG}} = -\frac{1}{2}\alpha\sum_j \frac{V_{0i}V_{0j}W_{0ij}}{\left|\boldsymbol{r}_{ij}^0\cdot\boldsymbol{r}_{ij}^0\right|}\left(E_i\delta_{ij}^i + E_j\delta_{ji}^j\right)\frac{\boldsymbol{r}_{ij}}{\left|\boldsymbol{r}_{ij}\right|}$$

- Hourglass mode → unphysical particle distribution

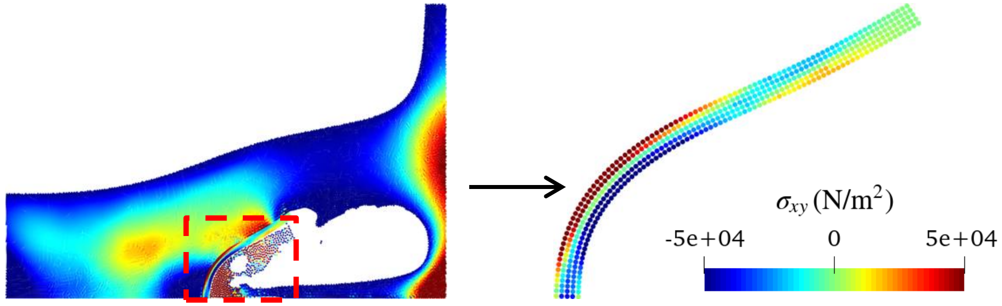
- Penalizes any local displacement not described by linear transformation of ***F***
- Parameter needs proper calibrations

[1] Ganzenmüller, Georg C. "An hourglass control algorithm for Lagrangian smooth particle hydrodynamics." *CMAME* (2015)

➢ Noises in fluid field



Simulated by δ-SPH - TLSPH



■ Velocity divergence errors → pressure noises → FSI coupling → stress noises

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2 Enhanced model : Overview

TLSPH structure model

$$\frac{D\boldsymbol{u}_i}{Dt}=\frac{1}{\rho_i^0}\sum_j\left(-\boldsymbol{P}_j\cdot\begin{bmatrix}\boldsymbol{A}_j^{0,1}\\ \boldsymbol{A}_j^{0,2}\end{bmatrix}\cdot\boldsymbol{M}_{ji}^0+\boldsymbol{P}_i\cdot\begin{bmatrix}\boldsymbol{A}_i^{0,1}\\ \boldsymbol{A}_i^{0,2}\end{bmatrix}\cdot\boldsymbol{M}_{ij}^0\right)\frac{\partial W_{ij}^0}{\partial r_{ij}^0}\frac{V_j^0}{|\boldsymbol{r}_{ij}^0|}$$

Coupling

F2nd, DHGC, RS

VEM-HPDC

δ-SPH fluid model

$$\frac{D\rho_i}{Dt}=-\rho_i\sum_j\boldsymbol{u}_{ij}\cdot\nabla_i W_{ij}V_j+D_i$$

$$\frac{D\boldsymbol{u}_i}{Dt}=-\sum_j m_j\left(\frac{p_i+p_j}{\rho_i\rho_j}\right)\nabla_i W_{ij}+\sum_j m_j\left(\frac{4\nu\boldsymbol{r}_{ij}\cdot\nabla_i W_{ij}}{(\rho_i+\rho_j)(\boldsymbol{r}_{ij}^2+\eta^2)}\right)\boldsymbol{u}_{ij}$$

$$p_i=c_f^2(\rho_i-\rho_0)$$

Enhanced schemes

(1) F2nd (Second-order deformation gradient tensor $\boldsymbol{F}$)

$$\boldsymbol{F}_i^S=\sum_j\frac{\boldsymbol{r}_{ij}}{|\boldsymbol{r}_{ij}^0|}\begin{bmatrix}\boldsymbol{A}_i^{0,1}\\ \boldsymbol{A}_i^{0,2}\end{bmatrix}\cdot\boldsymbol{M}_{ij}^0V_j^0\frac{\partial W_{ij}^0}{\partial r_{ij}^0}$$

DualSPHysics

(2) DHGC (**D**ynamic **H**ourglass **C**ontrol)

$$\boldsymbol{f}_i^{\mathrm{HG}}=-\frac{1}{2}\sum_j\frac{\alpha_i^k+\alpha_j^k}{2}\frac{V_{0i}V_{0j}W_{0ij}}{|\boldsymbol{r}_{ij}^0\cdot\boldsymbol{r}_{ij}^0|}\left(E_i\delta_{ij}^a+E_j\delta_{ji}^b\right)\frac{\boldsymbol{r}_{ij}}{|\boldsymbol{r}_{ij}|}$$

$$\alpha_i^k=|\boldsymbol{\varepsilon}_i|^k/|\boldsymbol{\varepsilon}_i|^{k-N}$$

$$\boldsymbol{\varepsilon}_{ij}^i=\langle\boldsymbol{r}_{ij}\rangle^i-\boldsymbol{r}_{ij}=\boldsymbol{F}_i\boldsymbol{r}_{ij}^0-\boldsymbol{r}_{ij}$$

(3) RS (**R**iemann SPH-based **S**tabilization term)

$$\frac{D\boldsymbol{u}_i}{Dt}=\frac{1}{\rho_i^0}\sum_j\left(-\boldsymbol{P}_j\cdot\begin{bmatrix}\boldsymbol{A}_j^{0,1}\\ \boldsymbol{A}_j^{0,2}\end{bmatrix}\cdot\boldsymbol{M}_{ji}^0+\boldsymbol{P}_i\cdot\begin{bmatrix}\boldsymbol{A}_i^{0,1}\\ \boldsymbol{A}_i^{0,2}\end{bmatrix}\cdot\boldsymbol{M}_{ij}^0\right)\frac{\partial W_{ij}^0}{\partial r_{ij}^0}\frac{V_j^0}{|\boldsymbol{r}_{ij}^0|}+\boldsymbol{\Pi}_i^{\mathrm{RS}}$$

(4) VEM-HPDC (**V**elocity divergence **E**rror **M**itigating and **H**yperbolic/**P**arabolic **D**ivergence **C**leaning schemes)

$$\frac{D\boldsymbol{u}_i}{Dt}=-2\sum_j m_j\left(\frac{p_i+p_j}{\rho_i\rho_j}\right)\nabla_i W_{ij}+\boldsymbol{g}+\boldsymbol{a}_i^{\mathrm{VEM}}-\nabla\psi_i$$

(1) F2nd: Improving accuracy of stress and strain computation

(2) DHGC: Suppressing zero-energy modes and dynamically adjusting hourglass control coefficient

(3) RS: Reducing high-frequency structure stress noises

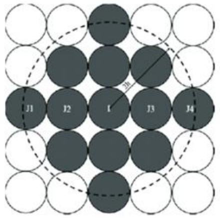
(4) VEM-HPDC: Mitigating velocity divergence errors and fluid pressure noises

2.1 Enhanced structure model : Second-order accuracy

Traditional first-order discretization (F1st)

$$\frac{D\boldsymbol{u}_i}{Dt} = \sum_j (\boldsymbol{P}_i L_i + \boldsymbol{P}_j L_j)\nabla_i W_{ij} V_i^0 V_j^0$$

$$\frac{D\boldsymbol{u}}{Dt} = \frac{1}{\rho_0}\nabla_0 \cdot \boldsymbol{P} + \boldsymbol{g}$$



$$\boldsymbol{F} = \nabla_0 \boldsymbol{r}$$

Taylor series

Second-order discretization (F2nd)

$$\frac{D\boldsymbol{u}_i}{Dt} = \frac{1}{\rho_i^0}\sum_j \left(-\boldsymbol{P}_j \cdot \begin{bmatrix} A_j^{0,1} \\ A_j^{0,2} \\ A_j^{0,3} \end{bmatrix} \cdot \boldsymbol{M}_{ji}^0 + \boldsymbol{P}_i \cdot \begin{bmatrix} A_i^{0,1} \\ A_i^{0,2} \\ A_i^{0,3} \end{bmatrix} \cdot \boldsymbol{M}_{ij}^0 \right) \frac{\partial W_{ij}^0}{\partial r_{ij}^0} \frac{V_j^0}{|\boldsymbol{r}_{ij}^0|}$$

$$\begin{bmatrix} A_i^{0,1} \\ A_i^{0,2} \\ \vdots \\ \vdots \\ A_i^{0,9} \end{bmatrix} = \begin{bmatrix} (M_{ij}^{0,1}, M_{ij}^{0,1}) & (M_{ij}^{0,1}, M_{ij}^{0,2}) & \cdots \cdots & (M_{ij}^{0,1}, M_{ij}^{0,9}) \\ (M_{ij}^{0,2}, M_{ij}^{0,1}) & (M_{ij}^{0,2}, M_{ij}^{0,2}) & & \vdots \\ \vdots & & \ddots & \vdots \\ \vdots & & \ddots & \vdots \\ (M_{ij}^{0,9}, M_{ij}^{0,1}) & \cdots & \cdots \cdots & (M_{ij}^{0,9}, M_{ij}^{0,9}) \end{bmatrix}^{-1}$$

$$\boldsymbol{M}_{ij}^0 = \left[\frac{x_{ij}^0}{|\boldsymbol{r}_{ij}^0|} \; \frac{y_{ij}^0}{|\boldsymbol{r}_{ij}^0|} \; \frac{z_{ij}^0}{|\boldsymbol{r}_{ij}^0|} \; \frac{1}{d_0}\frac{x_{ij}^0 x_{ij}^0}{|\boldsymbol{r}_{ij}^0|} \; \frac{1}{d_0}\frac{y_{ij}^0 y_{ij}^0}{|\boldsymbol{r}_{ij}^0|} \; \frac{1}{d_0}\frac{z_{ij}^0 z_{ij}^0}{|\boldsymbol{r}_{ij}^0|} \; \frac{1}{d_0}\frac{x_{ij}^0 y_{ij}^0}{|\boldsymbol{r}_{ij}^0|} \; \frac{1}{d_0}\frac{x_{ij}^0 z_{ij}^0}{|\boldsymbol{r}_{ij}^0|} \; \frac{1}{d_0}\frac{y_{ij}^0 z_{ij}^0}{|\boldsymbol{r}_{ij}^0|} \right]^T$$

- Coefficient matrix ***A*** is computed at the beginning

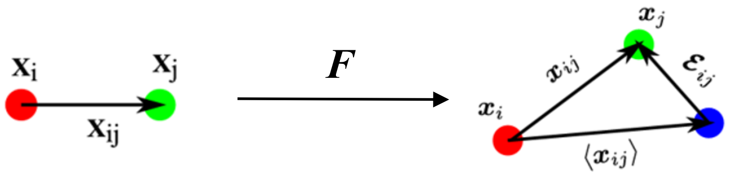
- 1st Order → 2nd Order, without reducing much computational efficiency
- The first three-dimensional 2nd order accuracy GPU-based structure solver

[1] Khayyer et al. "An improved Riemann SPH-Hamiltonian SPH coupled solver for hydroelastic fluid-structure interactions." *EABE* (2024)

2.2 Enhanced structure model : Hourglass mode control

➤ Dynamic hourglass control (DHGC)

[1]



Traditional hourglass control scheme (HGC)

$$\boldsymbol{f}_i^{HG} = -\frac{1}{2}\alpha \sum_j \frac{V_{0i} V_{0j} W_{0ij}}{\left|\boldsymbol{r}_{ij}^0 \cdot \boldsymbol{r}_{ij}^0\right|}\left(E_i \delta_{ij}^i + E_j \delta_{ji}^j\right)\frac{\boldsymbol{r}_{ij}}{\left|\boldsymbol{r}_{ij}\right|} \quad \delta_{ij}^i = \frac{\epsilon_{ij}^i \cdot \boldsymbol{r}_{ij}}{r_{ij}} \text{ with } \epsilon_{ij}^i = \left\langle \boldsymbol{r}_{ij} \right\rangle^i - \boldsymbol{r}_{ij} = \boldsymbol{F}_i \boldsymbol{r}_{ij}^0 - \boldsymbol{r}_{ij}$$

- α is a constant and needs proper calibrations

Dynamic hourglass control scheme (DHGC)

$$\boldsymbol{f}_i^{HG} = -\frac{1}{2}\sum_j \frac{\alpha_i^k + \alpha_j^k}{2} \frac{V_{0i} V_{0j} W_{0ij}}{\left|\boldsymbol{r}_{ij}^0 \cdot \boldsymbol{r}_{ij}^0\right|}\left(E_i \delta_{ij}^a + E_j \delta_{ji}^b\right)\frac{\boldsymbol{r}_{ij}}{\left|\boldsymbol{r}_{ij}\right|} \qquad \alpha_i^k = \frac{\left|\epsilon_i\right|^k}{\left|\epsilon_i\right|^{k-N}}$$

- Adaptively adjust based on error vector

➤ Riemann Stabilization (RS)

$$\frac{D\boldsymbol{u}_i}{Dt} = \frac{1}{\rho_i^0}\sum_j \left(-\boldsymbol{P}_j \cdot \begin{bmatrix} \boldsymbol{A}_j^{0,1} \\ \boldsymbol{A}_j^{0,2} \end{bmatrix} \cdot \boldsymbol{M}_{ji}^0 + \boldsymbol{P}_i \cdot \begin{bmatrix} \boldsymbol{A}_i^{0,1} \\ \boldsymbol{A}_i^{0,2} \end{bmatrix} \cdot \boldsymbol{M}_{ij}^0 \right) \frac{\partial W_{ij}^0}{\partial r_{ij}^0} \frac{V_j^0}{\left|\boldsymbol{r}_{ij}^0\right|} + \boldsymbol{\Pi}_i^{RS}$$

$$\boldsymbol{\Pi}_i^{RS} = \sum_j \boldsymbol{P}_{ij}^{RS} \cdot \nabla_0 W_{ij} V_j^0, \ \boldsymbol{P}_{ij}^{RS} = \det\left(\boldsymbol{F}_i\right) \Pi_{ij}^{RS} \boldsymbol{F}_i^{-T}$$

$$\Pi_{ij}^{RS} = \frac{\beta c_0}{\rho_i} \frac{\rho_i + \rho_j}{2} \frac{\boldsymbol{u}_{ij} \cdot \boldsymbol{r}_{ij}}{\left|\boldsymbol{r}_{ij}\right|}; \beta = \max\left(0, \frac{u_L - u_R}{\left|u_L - u_R\right|}\right)$$

$$u_L = \boldsymbol{u}_i \cdot \frac{\boldsymbol{r}_{ij}}{\left|\boldsymbol{r}_{ij}\right|}; u_R = \boldsymbol{u}_j \cdot \frac{\boldsymbol{r}_{ij}}{\left|\boldsymbol{r}_{ij}\right|}$$

- Reduce noises in stress field
- No artificial parameters

[1] Ganzenmüller, Georg C. "An hourglass control algorithm for Lagrangian smooth particle hydrodynamics." *CMAME* (2015)
[2] Khayyer et al. "An improved updated Lagrangian SPH method for structural modelling." *CPM* (2024)

2.3 Enhanced fluid model : Velocity divergence clean

➢ Velocity-divergence Error Mitigating (VEM) [1]

$$p_a^{\mathrm{VEM}} = c_s^2 d\rho_a = c_s^2 \Delta t \left(\frac{D\rho}{Dt}\right)_a^{k-1} = -\rho_a c_s^2 \Delta t \langle \nabla \cdot \boldsymbol{u} \rangle_a^{k-1}$$

Pressure related to velocity divergence error ⇩ This error will be accumulated

$$\boldsymbol{a}_a^{\mathrm{VEM}} = -\frac{1}{\rho_a}\sum_b F\left(p_a^{\mathrm{VEM}}, p_b^{\mathrm{VEM}}\right)\nabla_a W_{ab} V_b$$

$$F(p_a, p_b) = \begin{cases} p_b + p_a & (p_a \geq 0 \cup a \notin \Omega_{IN}) \\ p_b - p_a & (p_a < 0 \cap a \in \Omega_{IN}) \end{cases}$$

$$\frac{D\boldsymbol{u}_a}{Dt} = -\sum_b m_b \left(\frac{F(p_a, p_b)}{\rho_a \rho_b} + \Pi_{ab}\right)\nabla_a W_{ab} + \boldsymbol{g}_a + \boldsymbol{a}_a^{\mathrm{VEM}}$$

➢ Hyperbolic/Parabolic Divergence Cleaning (HPDC) [2]

$$\begin{cases} \left(\frac{D\boldsymbol{u}}{Dt}\right)_\psi + \nabla \psi = 0 \\ \mathcal{D}(\psi) + \nabla \cdot \boldsymbol{u} = 0 \end{cases}$$

Hyperbolic term ⇩ Parabolic term

$$\frac{\partial^2 (\nabla \cdot \boldsymbol{u})}{\partial t} - c_h^2 \nabla^2 (\nabla \cdot \boldsymbol{u}) + \frac{c_h^2}{c_p^2}\frac{\partial (\nabla \cdot \boldsymbol{u})}{\partial t} = 0$$

⇩

$$\frac{D\boldsymbol{u}_a}{Dt} = -2\sum_b m_b \left(\frac{p^*}{\rho_a \rho_b}\right)\nabla_a W_{ab} + \boldsymbol{g} + \boldsymbol{a}_a^{\mathrm{VEM}} - \nabla \psi_a$$

$\psi = 0$ for boundary particles

[1] Khayyer et al. "Enhanced resolution of the continuity equation in explicit weakly compressible SPH simulations of incompressible free-surface fluid flows." *AMM* (2023)

[2] Fourtakas et al. "Divergence cleaning for weakly compressible smoothed particle hydrodynamics." *C&F* (2025)

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3 Numerical validations

➢ Five cases

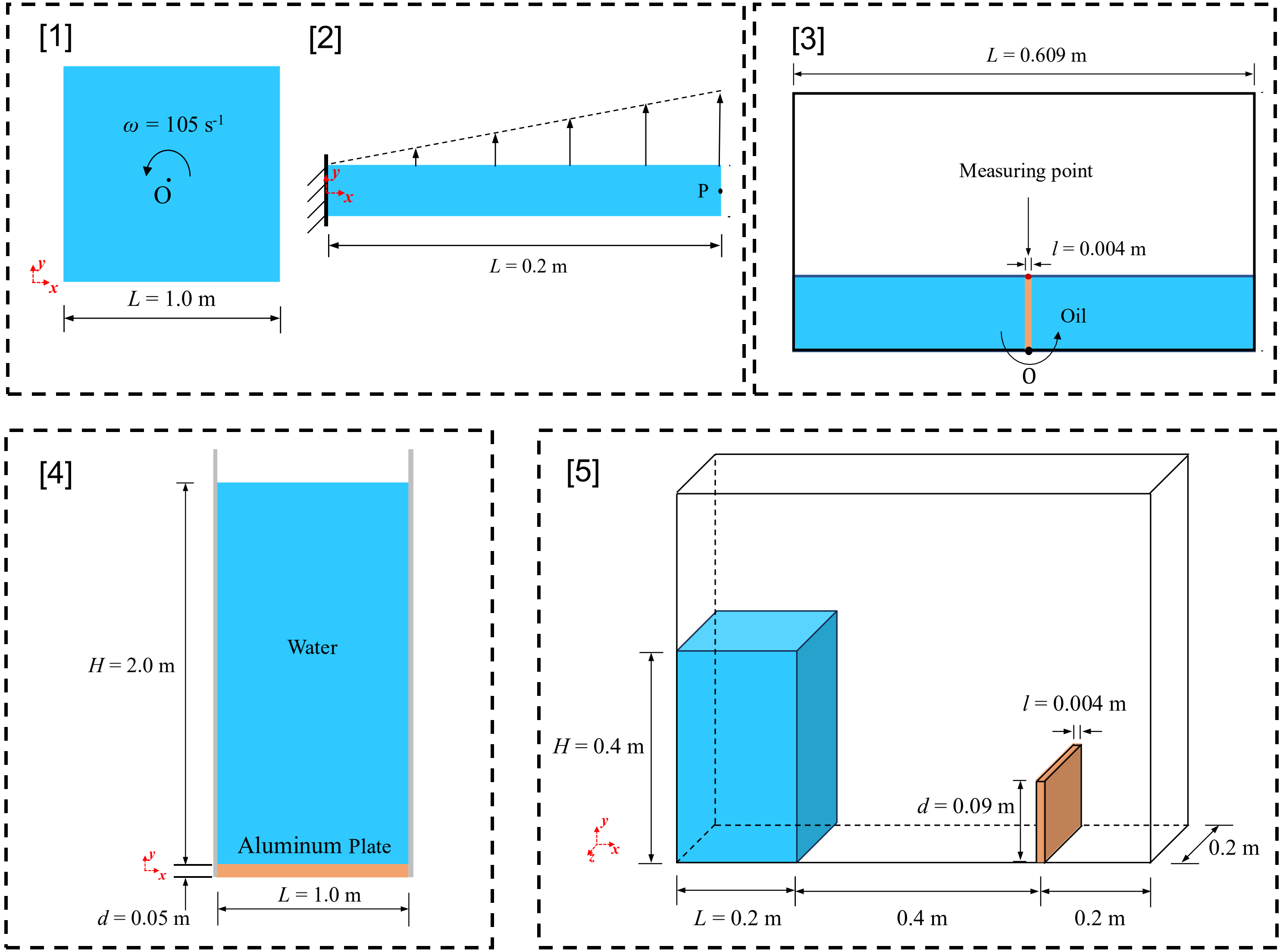
[1] High speed rotation of elastic square

[2] Free oscillation of a cantilever plate

[3] Liquid sloshing with an elastic plate

[4] Hydrostatic water column

[5] Dam break with an elastic plate



References

[1] Lee, C. H, et al. " Development of a cell centred upwind finite volume algorithm for a new conservation law formulation in structural dynamics. " *C&S* (2013)

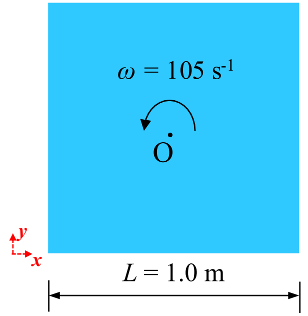
[2] Gray, James P., et al. "SPH elastic dynamics." *CMAME* (2001)

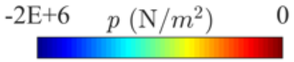
[3] Idelsohn, S. R., et al. "Interaction between an elastic structure and free-surface flows: experimental versus numerical comparisons using the PFEM." *CM* (2008)

[4] Fourey, G., et al. "An efficient FSI coupling strategy between smoothed particle hydrodynamics and finite element methods." *CPC* (2017)

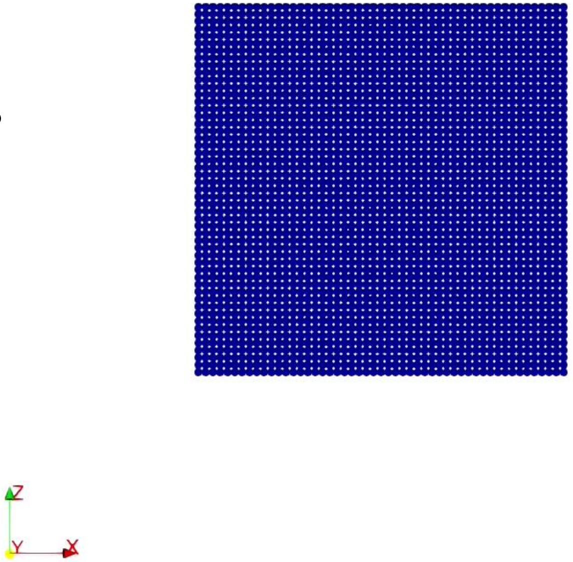
[5] Liao, et al. "Free surface flow impacting on an elastic structure: Experiment versus numerical simulation." *APOR* (2015)

3.1 High speed rotation of elastic square: Pressure field

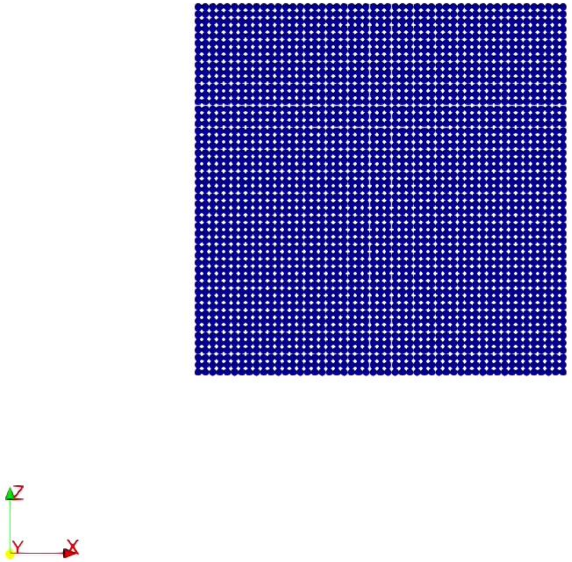




Original TLSPH model



Present model



L = 1 m, dp = 0.01 m

E = 2 Mpa, ρ = 1000 kg/m^3

v = 0.49, C_s = 242 m/s

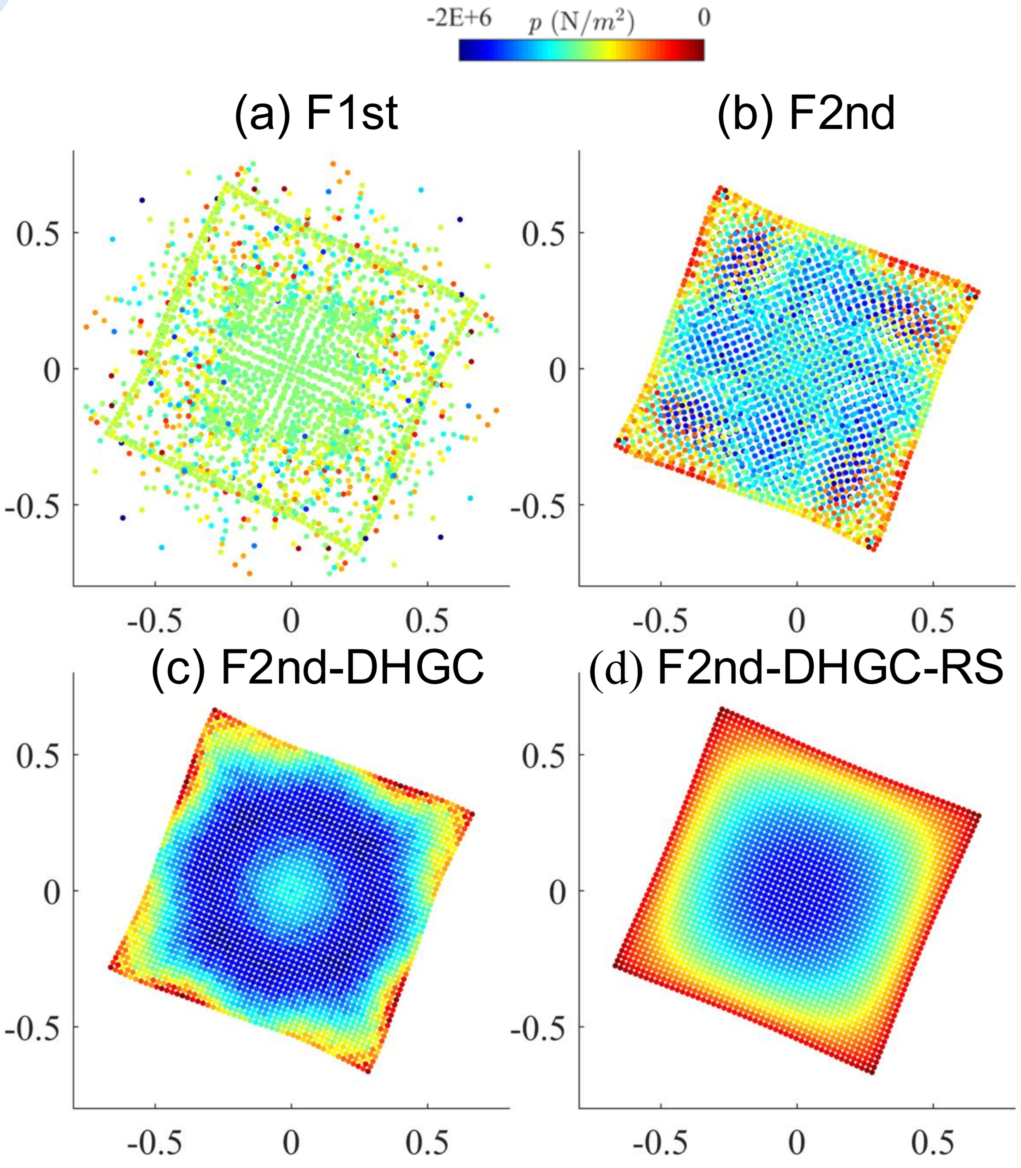
Predictor-Corrector,

C_{CFL} = 0.2

5th Wendland C2, h/dp = 2

- Improves accuracy, reduces noises and effectively suppresses hourglass modes

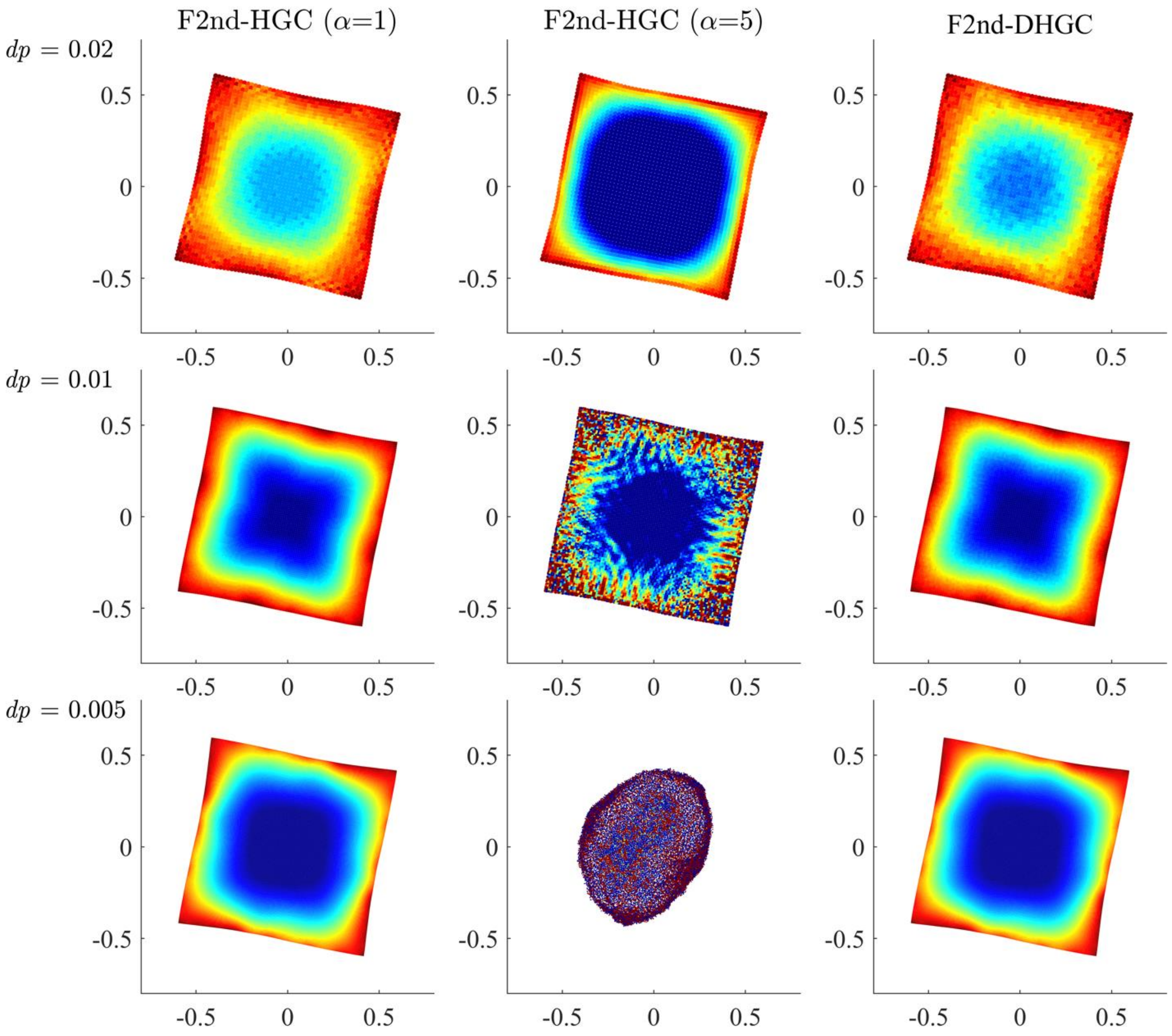
3.1 High speed rotation of elastic square: Pressure field



- F2nd → improves the solution accuracy
- DHGC → mitigates hourglass modes
- RS → reduces stress noises

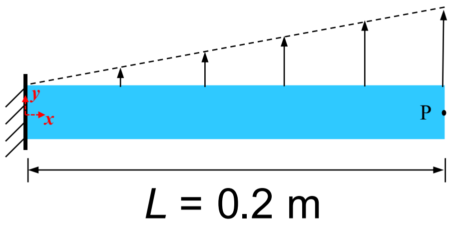
3.1 High speed rotation of elastic square: Pressure field

➢ Convergence analysis



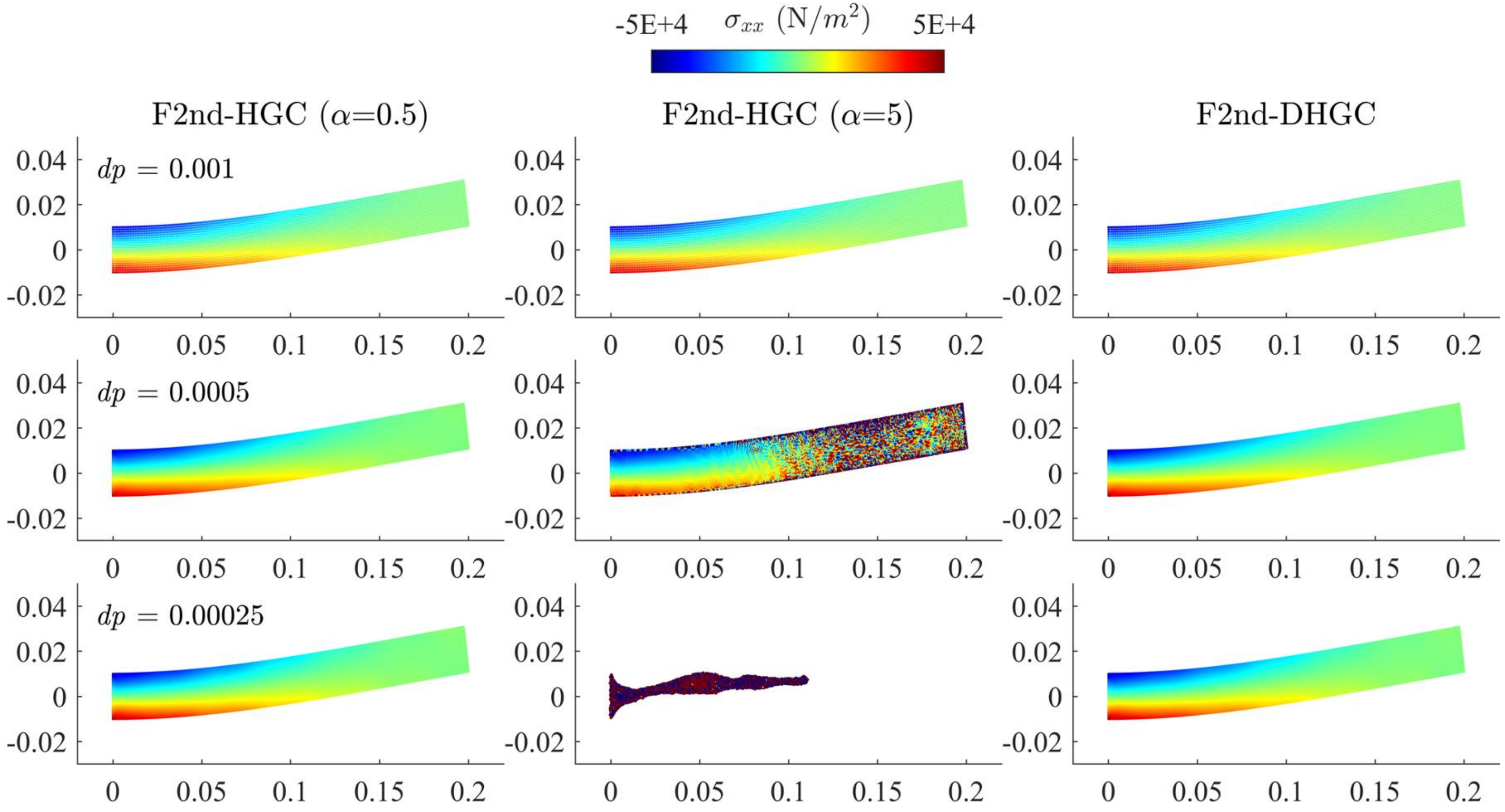
- HGC: noticeable unphysical stress fluctuations (α = 5) at dp = 0.01, 0.005
- DHGC: resolution-independent

3.2 Free oscillation of a cantilever plate: Stress field



E = 2 Mpa, ρ = 1000 kg/m^3 v = 0.3975

$$\begin{cases} u_y(x) = \xi c_s \dfrac{f(x)}{f(L)} \\ f(x) = (\cos kL + \cosh kL)(\cosh kx - \cos kx) + (\sin kL - \sinh kL)(\sinh kx - \sin kx) \end{cases}$$

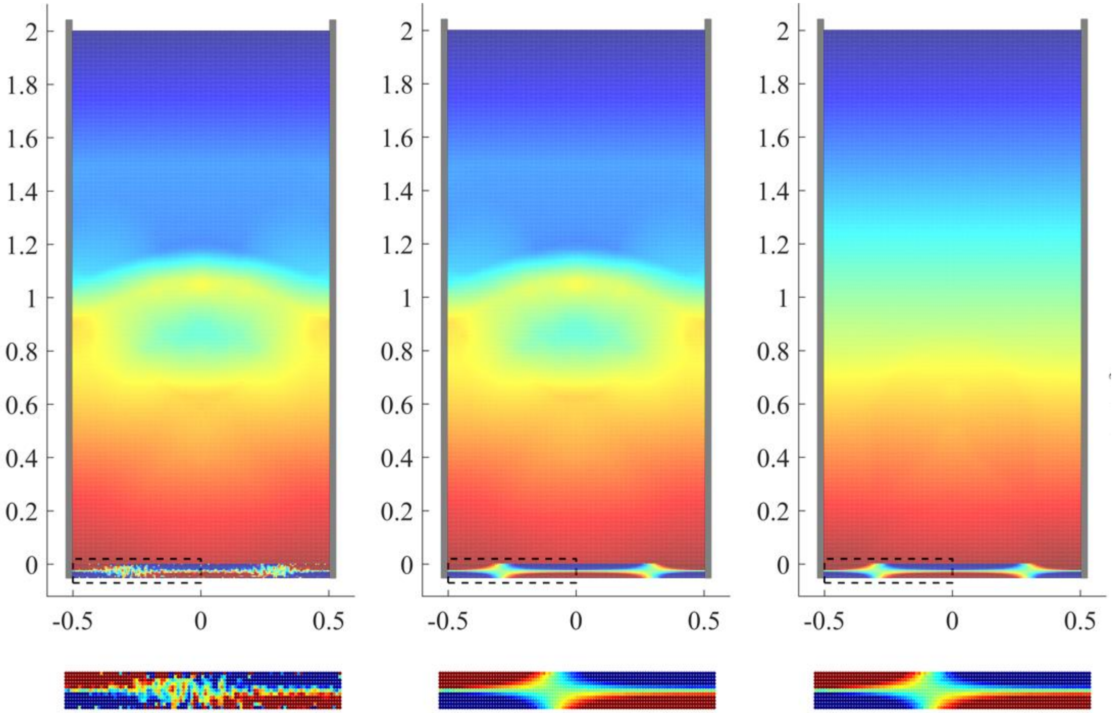


- HGC (α = 5) → excessive hourglass control → over-stiffness
- DHGC → Dynamically adjusted based on the errors

3.3 Hydrostatic water column & Sloshing

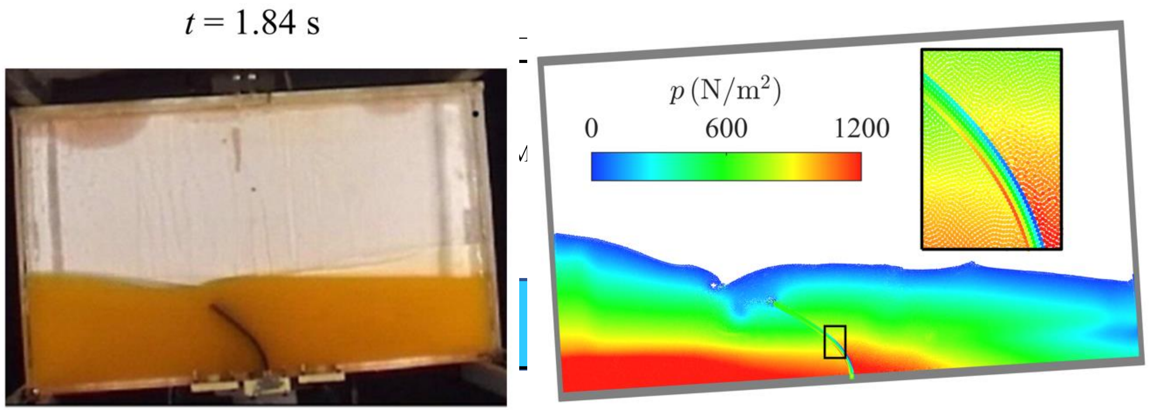
E = 67.5 Gpa, ρ = 2700 kg/m³ v = 0.34

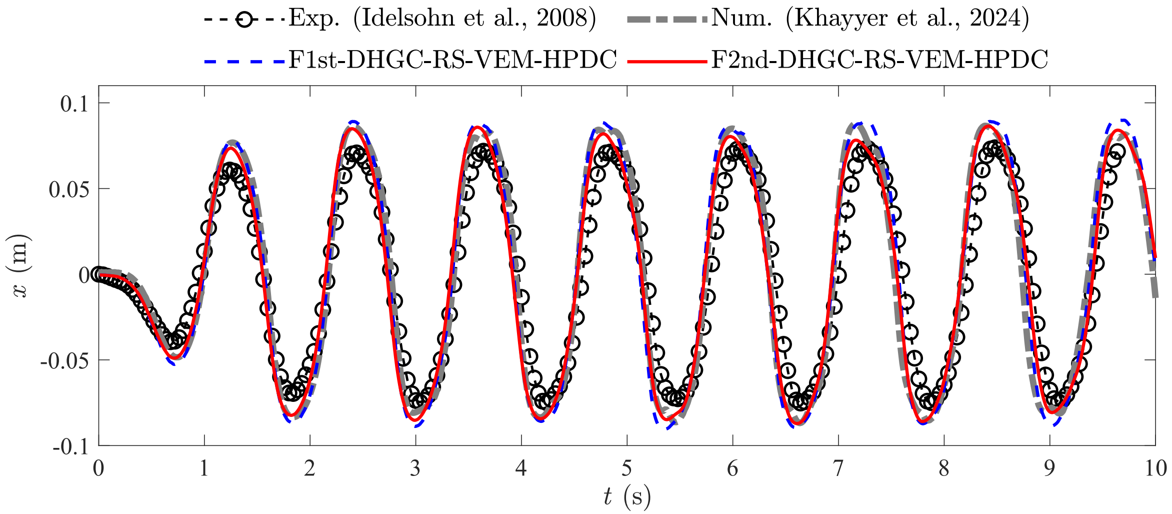
F2nd | F2nd-RS | F2nd-RS-VEM-HPDC



- Reduces noises in both fluid and structure fields

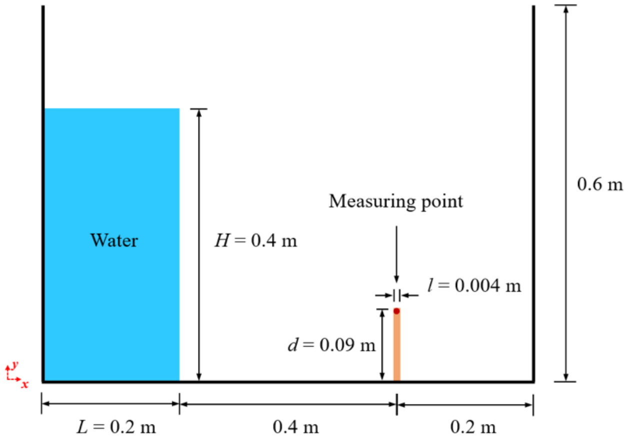
E = 6 Mpa, ρ = 1100 kg/m³, v = 0.49





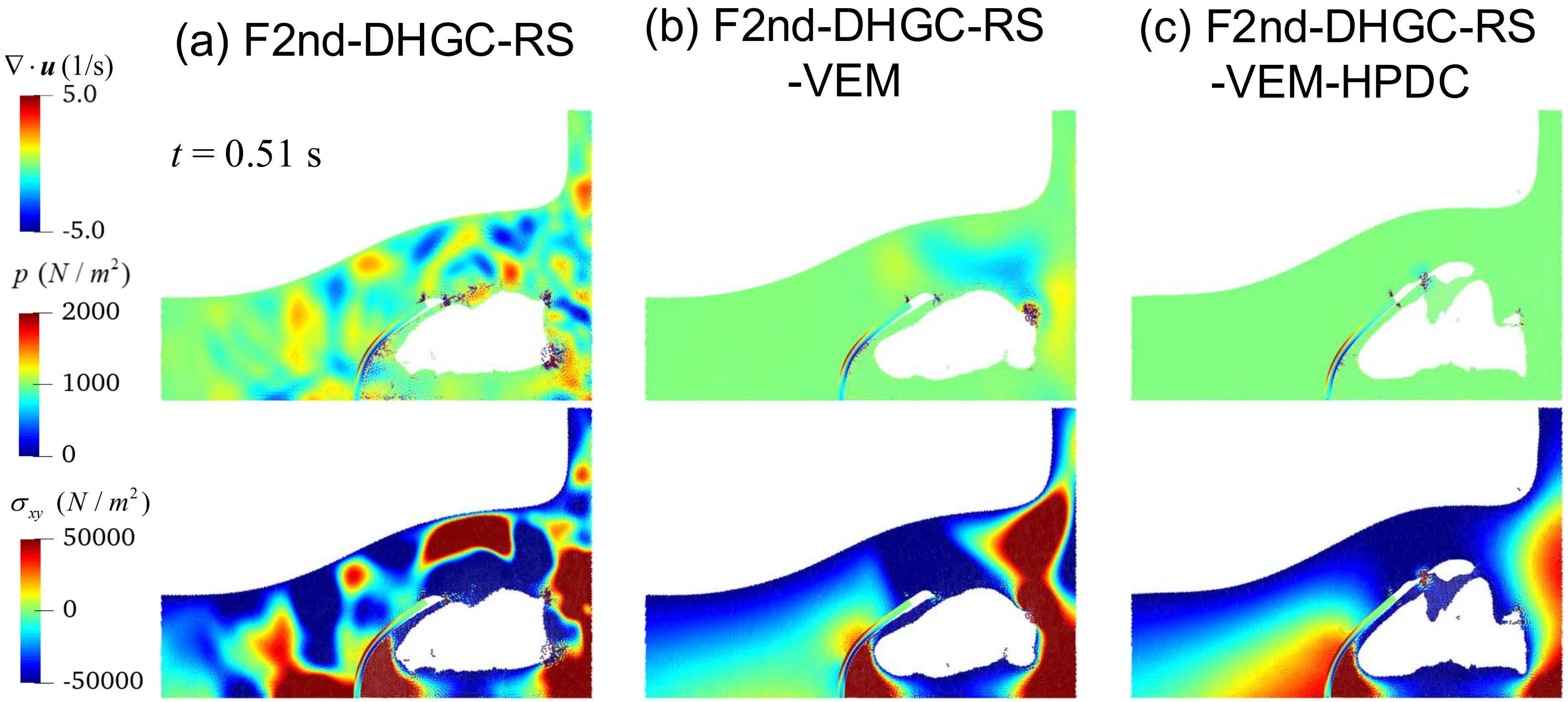
- The displacement history predicted by the present model is closer to the experimental result

3.4 Dam break with elastic plate : Pressure and velocity divergence fields



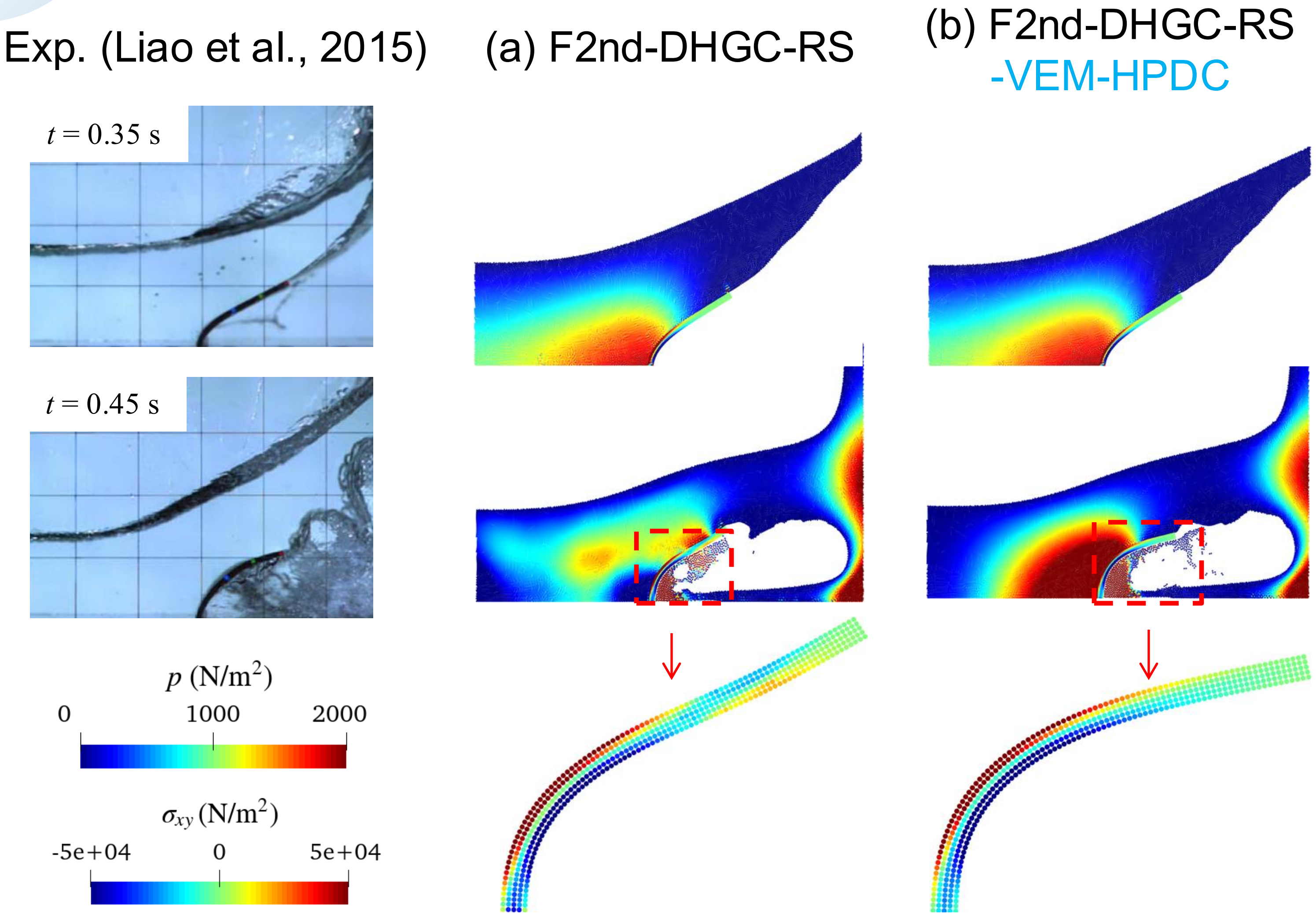
E = 12 Mpa

ρ = 1161.5 kg/m^3 v = 0.4



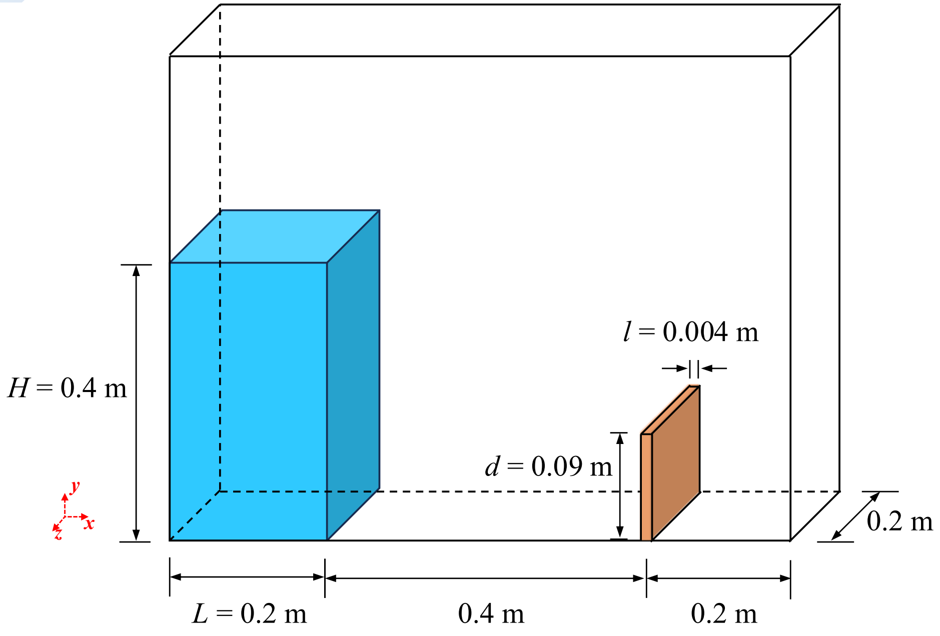
- VEM, HPDC → Reduces velocity divergence error and enhances pressure field

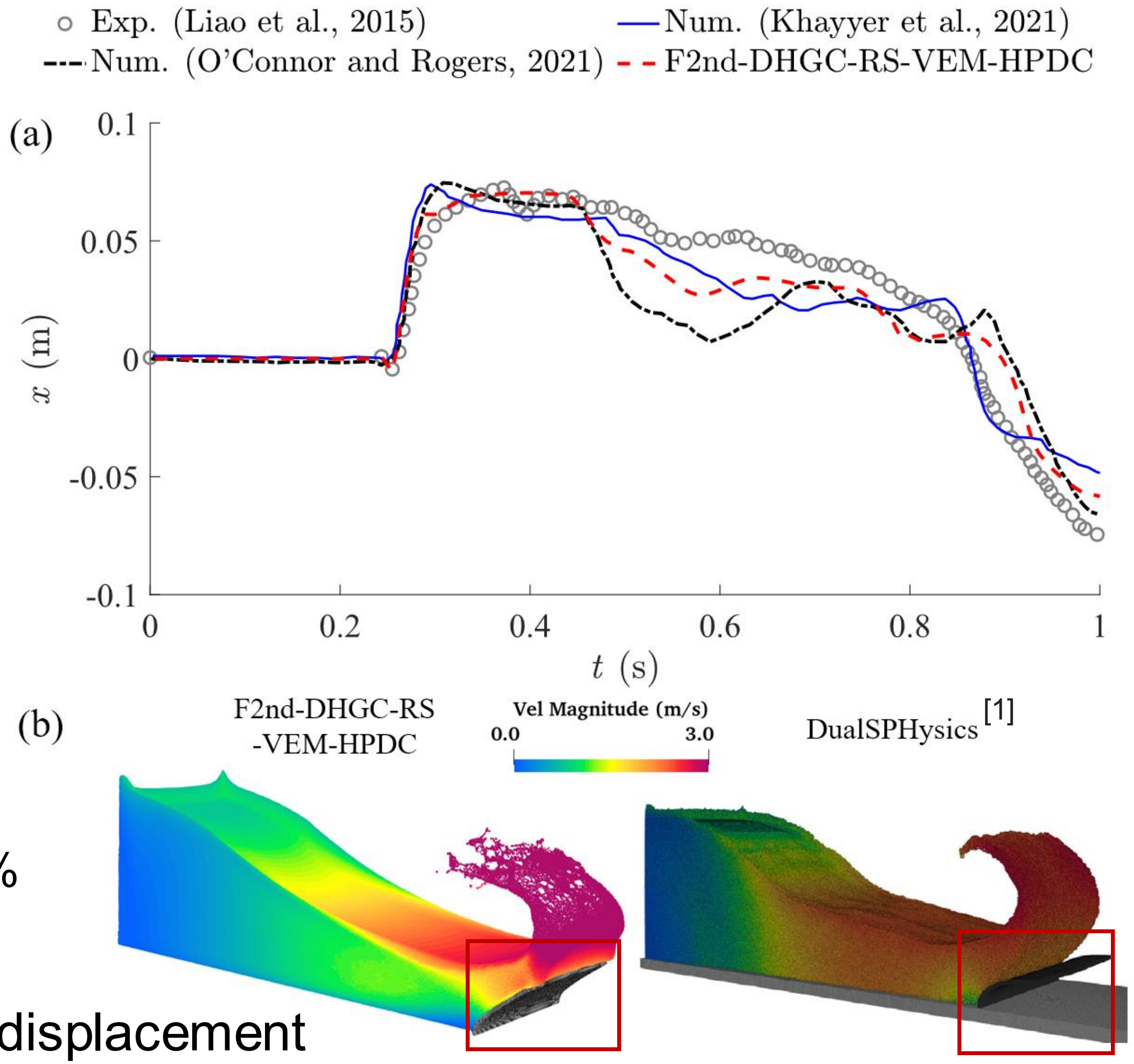
3.4 Dam break with elastic plate : Pressure and stress fields



- Satisfactory agreement with the experimental photos
- Divergence cleaning → reduces fluid pressure noises → enhances the fluid-structure coupling → reduces stress noises

3.4 Dam break with elastic plate : 3D simulation





dp = 0.001 m, *t* = 1 s, *N* ≈ 24,000,000, ≈ 60 hours

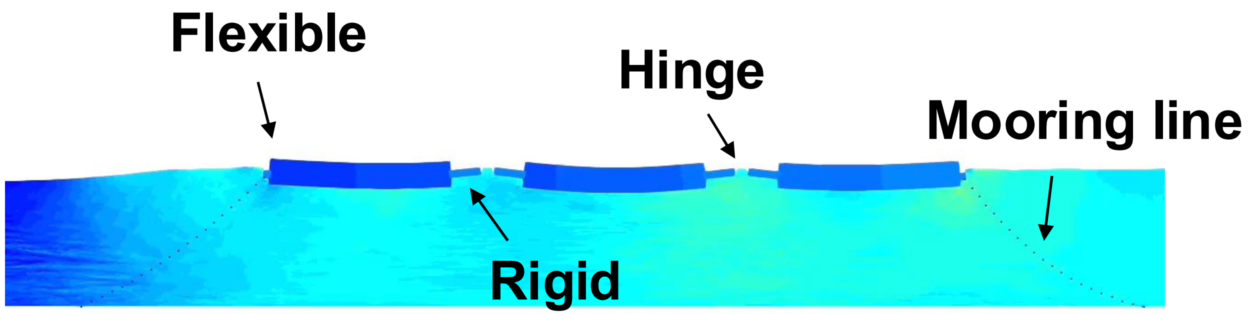
Computational time and memory increases ≈ 20% and 25%

- Improved accuracy in predicting deformation and displacement

[1] O'Connor and Rogers. "A fluid–structure interaction model for free-surface flows and flexible structures using smoothed particle hydrodynamics on a GPU." *JFS* (2021)

4 Summary: Enhanced FSI model

- F2nd → **Improve solution accuracy** of TLSPH
- DHGC → **Mitigate hourglass modes** and being **case- and resolution-independent**
- RS → **Reduce noises in structure stress field** and being **parameter free**
- VEM, HPDC → **Reduce noises in fluid pressure field**, enhancing the fluid structure coupling
- On going/ future work → Engineering applications: Flexible floating arrays



Thank you for your attention